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Painlevé property and pseudopotentials for non-linear evolution equations

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Abstract. In this paper, the link between the singularity manifold equation and the pseudopotential equations is found for non-linear evolution equations both in 1+1 and 2+1 dimensions. Many examples are given: Burgers, Korteweg–de Vries, modified Korteweg–de Vries, Harry Dym, a higher-order Korteweg–de Vries, Sawada–Kotera, Kaup–Kuperschmidt, Kadomtsev–Petviashvili, Boussinesq, modified Kadomtsev–Petviashvili, and $(2+1)$-dimensional Sawada–Kotera equations.

1. Introduction

In Weiss *et al* (1983) the Painlevé property for partial differential equations was established. Since then many papers have been dedicated to its applications. Weiss himself applied the Painlevé analysis to various equations (Weiss 1983, 1984, 1985a, b). In Nucci (1988a, b) it was shown that if the equations satisfied by the pseudopotential (Wahlquist and Estabrook 1975) are of a Riccati type, then one can easily obtain both the Lax equations and the auto-Bäcklund transformation for the corresponding non-linear evolution equation in both 1+1 and 2+1 dimensions. In the present paper, we show how to obtain the singularity manifold equation. The paper is organised as follows. In §2 the general idea is established through the properties of the Riccati equation. In §3 several examples in 1+1 dimensions are given: Burgers, Korteweg–de Vries (KdV), modified KdV, Harry Dym, higher-order KdV, Sawada–Kotera (SK), and Kaup–Kupershmidt (KK) equations. In §4 there are some examples in 2+1 dimensions: Kadomtsev–Petviashvili (KP), Boussinesq, modified KP, and $(2+1)$-dimensional SK equations.

2. Riccati equation and integrability properties

Figure 1 shows well known properties of the Riccati equation (Hille 1976) on the dependent variable v and independent variable x: the cross ratio of four solutions v_i $(i = 1,2,3,4)$ is a constant; it is formally invariant under a Möbius transformation (a,b,c,d = constants, $ad - bc \neq 0$), i.e. another Riccati equation on the dependent variable v^* can be derived; there is a transformation to a linear second-order differential equation on the dependent variable r, which can be transformed into a third-order differential equation, on the dependent variable w, involving the Schwarzian derivative, i.e.

$$\{w;x\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2. \tag{2.1}$$

Note that in figure 1 the Riccati equation has the coefficient of u^2 equal to a constant k and the linear term is missing to apply the transformation which leads directly to the same third-order differential equation in w. Now, let us consider non-linear evolution equations in 1+1 dimensions:

$$q_t = H(q, q_x, q_{xx}, q_{xxx}, \ldots) \tag{2.2}$$

As in Nucci (1988a), we assume there exists a pseudopotential $u = u(x,t)$ such that

$$u_x = F_2(q)u^2 + F_1(q)u + F_0(q) \tag{2.3a}$$

$$u_t = G(u, q, q_x, q_{xx}, \ldots) \tag{2.3b}$$

with G a polynomial of second order in u. The system (2.3) is subject to the integrability condition $u_{xt} = u_{tx}$ whenever (2.2) is satisfied. If we can find a pseudopotential u for (2.2) such that F_2 is a constant, say k, then we obtain the Lax equations from (2.3) by means of:

$$u = -\frac{1}{k}(\ln \psi)_x \tag{2.4}$$

with ψ the spectral function. If the equation satisfied by u, obtained by eliminating q from (2.3), is invariant under the Möbius group, i.e. if

$$u^* = \frac{a + bu}{c + du} \tag{2.5}$$

then, combining

$$u^*_x = F_2(q^*)u^{*2} + F_1(q^*)u^* + F_0(q^*) \tag{2.6}$$

and (2.3a) by means of (2.5), we obtain the spatial part of an auto-Bäcklund transformation for (2.2), q^* being another solution of (2.2). The time part is obtained from (2.3b) in a similar way. In Anderson and Ibraghimov (1979), the recurrence of the cross ratio property both in Riccati equations and auto-Bäcklund transformations was remarked. Finally, we get the singularity manifold equation with dependent variable $\phi = \phi(x,t)$ through the Lax equations and

$$\phi = \psi_1/\psi_2 \tag{2.7}$$

with ψ_1 and ψ_2 being two linear independent solutions of the Lax equations. If there exists a pseudopotential u such that (2.3a) is given by

$$u_x = ku^2 + F_0(q) \tag{2.8}$$

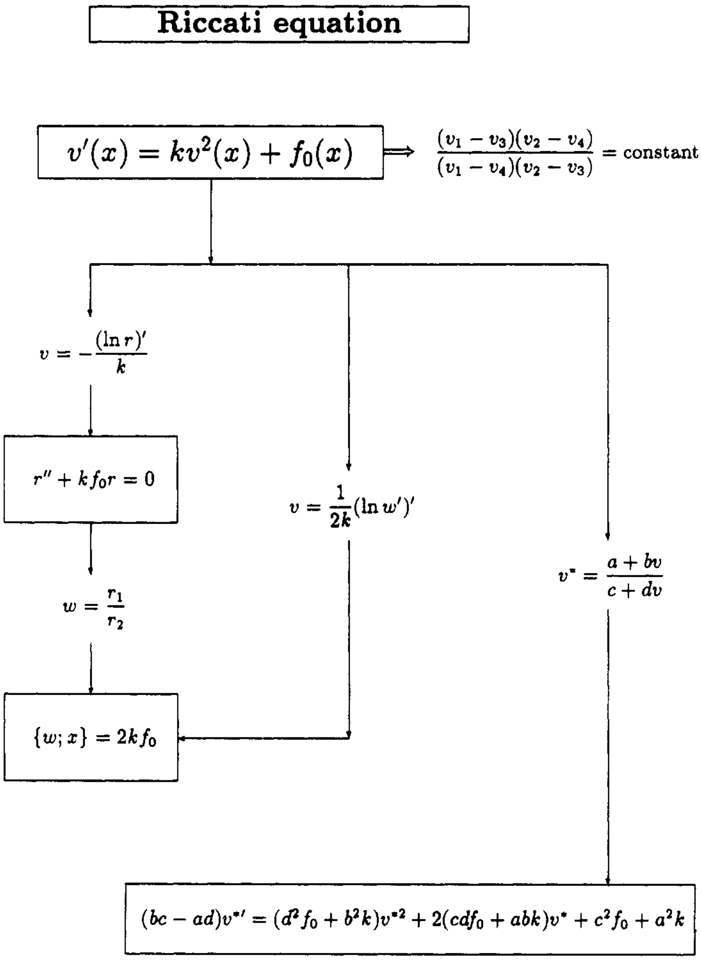

Figure 1. Properties of the Riccati equation.

then the singularity manifold equation is obtained directly from (2.3) through:

$$u = \frac{1}{2k}(\ln \phi_x)_x. \tag{2.9}$$

In this case (2.4) and (2.9) imply

$$\phi_x = \psi^{-2}. \tag{2.10}$$

The $(2+1)$-dimensional case is rather more complicated. Let us consider non-linear evolution equations in the form:

$$q_t = H\left(q, q_x, q_y, q_{xx}, q_{xy}, \int q_y \, dx, \ldots\right). \tag{2.11}$$

As in Nucci (1988b), we assume there exists a pseudopotential $u = u(x, y, t)$ such that:

$$u_x = F\left(u, q, \int u_y \, dx, \int u \, dx\right) \tag{2.12a}$$

$$u_t = \left[G\left(u, q, q_x, \int u_y \, dx, \ldots\right)\right]_x \tag{2.12b}$$

with the requirement $u_{xt} = u_{tx}$ whenever (2.11) is satisfied. If we can impose F to be a polynomial of second order in u with constant coefficient for u^2, say k, then (2.4) could be applied to (2.12), which will give the Lax equations for (2.11). If the equation satisfied by u is invariant under (2.5), then combining

$$u_x^* = F\left(u^*, q^*, \int u_y^* \, dx, \int u^* \, dx\right) \tag{2.13}$$

and (2.12a) by means of (2.5), we obtain the spatial part of an auto-Bäcklund transformation for (2.11), q^* being another solution of (2.11). Finally, we get the singularity manifold equation for (2.11) from the Lax equations through (2.7), as in 1+1 dimensions. It should be remarked that (2.10) is not true in 2+1 dimensions, because the Wronskian of ψ_1 and ψ_2 with respect to x is never equal to one.

3. Examples in 1+1 dimensions

Here the results of §2 are applied to some non-linear evolution equations in 1+1 dimensions: Burgers (figure 2), KdV (figure 3), modified KdV (figure 4), Harry Dym (figure 5), a higher-order KdV (figure 6), SK (figure 7) and KK (figure 8) equations. Their pseudopotentials represent a wide range of possible cases, because they have properties different from each other. In the cases of KdV, Harry Dym (for which we obtain a singularity manifold equation unknown in the literature) and a higher-order KdV equation, the singularity manifold equation is obtained by either (2.7) or (2.9). In the cases of Burgers and modified KdV equations, there is no pseudopotential such that (2.8) is true: their singularity manifold equations are obtained, respectively, by (2.7)

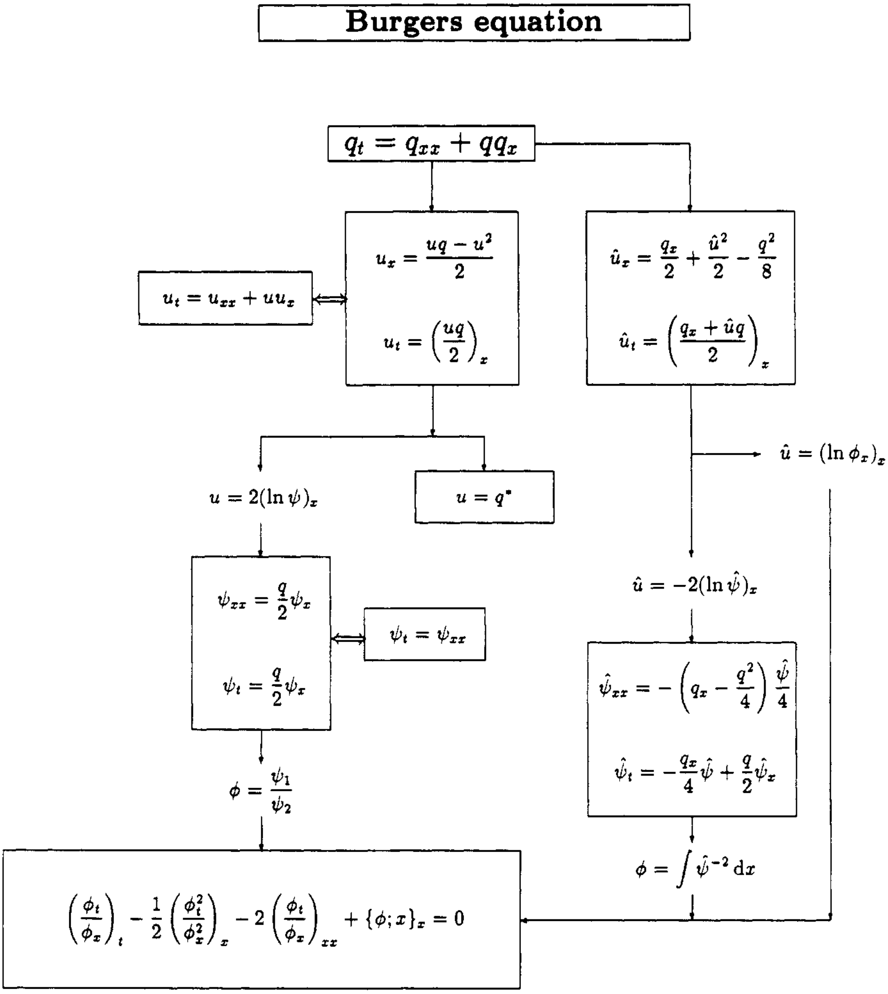

Figure 2. Properties of the Burgers equation. More details can be found in Nucci (1990).

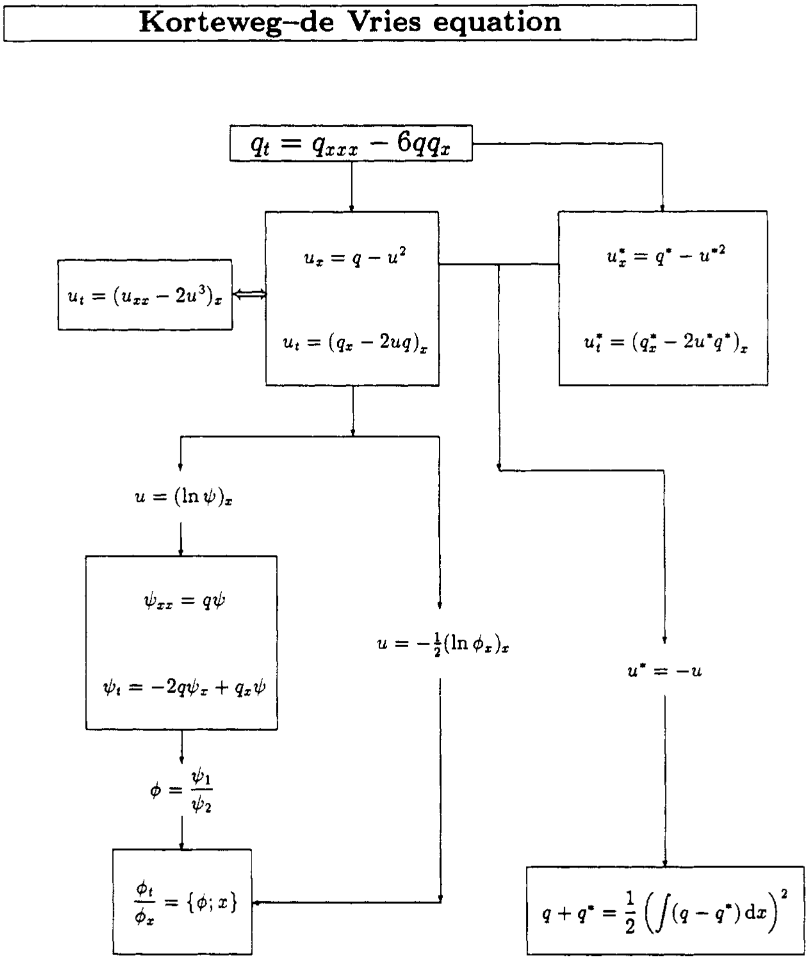

Figure 3. Properties of the Korteweg–de Vries equation. Note that the pseudopotential satisfies the modified KdV equation.

Modified Korteweg–de Vries equation

$$u_x = -2uq + u^2$$

$$u_t = (-2uq_x - 2uq^2)_x$$

$$u^* = -u$$

$$u^*_x = -2u^*q^* + u^{*2}$$

$$u^*_t = (-2u^*q^*_x - 2u^*q^{*2})_x$$

$$q - q^* = \exp\left(-\int (q + q^*)\, dx\right)$$

$$u = -(\ln \psi)_x$$

$$\hat{u}_t = [2(q_x - q^2)\hat{u} + q_{xx} - 2qq_x]_x$$

$$\hat{u} = \tfrac{1}{2}(\ln \psi_x)_x$$

$$\psi_{xx} = -2q\psi_x$$

$$\psi_t = -2(q_x + q^2)\psi_x$$

$$\frac{\psi_t}{\psi_x} = \{\psi; x\}$$

$$\hat{u} = -(\ln \hat{\psi})_x$$

$$\hat{\psi}_{xx} = -(q_x - q^2)\hat{\psi}$$

$$\hat{\psi}_t = 2(q_x - q^2)\hat{\psi}_x - (q_{xx} - 2qq_x)\hat{\psi}$$

Figure 4. Properties of the modified Korteweg–de Vries equation. Note that $\hat{\psi}$ is the spectral function, while ψ is the trivial one.

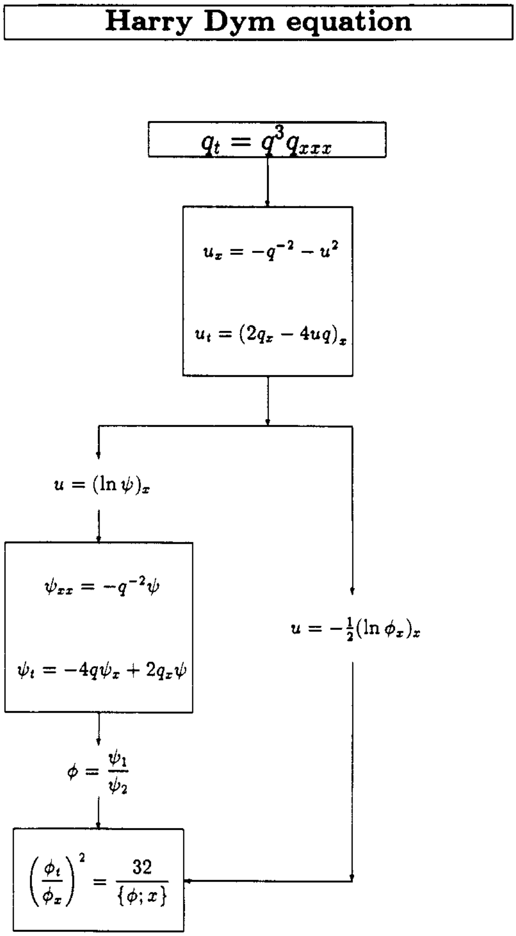

Figure 5. Properties of the Harry Dym equation. Note that although this equation does not possess the Painlevé property (Weiss 1983), a 'singularity manifold equation' is nevertheless obtained.

A higher-order Korteweg–de Vries equation

$$q_t = (q_{xxxx} + 5q_x^2 + 10qq_{xx} + 10q^3)_x$$

$$u_x = q + u^2$$

$$u_t = (q_{xxx} + 2uq_{xx} + 6qq_x + 6uq^2)_x$$

$$u_x^* = q^* + u^{*2}$$

$$u_t^* = (q_{xxx}^* + 2u^*q_{xx}^* + 6q^*q_x^* + 6u^*q^{*2})_x$$

$$u = -(\ln\psi)_x$$

$$\psi_{xx} = -q\psi$$

$$\psi_t = 2(q_{xx} + 3q^2)\psi_x - (q_{xxx} + 6qq_x)\psi$$

$$\phi = \frac{\psi_1}{\psi_2}$$

$$\frac{\phi_t}{\phi_x} = \{\phi; x\}_{xx} + \frac{3}{2}\{\phi; x\}^2$$

$$u = \tfrac{1}{2}(\ln\phi_x)_x$$

$$u^* = -u$$

$$q + q^* = -\frac{1}{2}\left(\int (q - q^*)\,\mathrm{d}x\right)^2$$

Figure 6. Properties of the higher-order Korteweg–de Vries equation. Morris (1977) used the differential form approach (Corones and Testa 1976) to generate the Lax equations. All the above results can be easily obtained for the entire KdV hierarchy (Rogers and Nucci 1986).

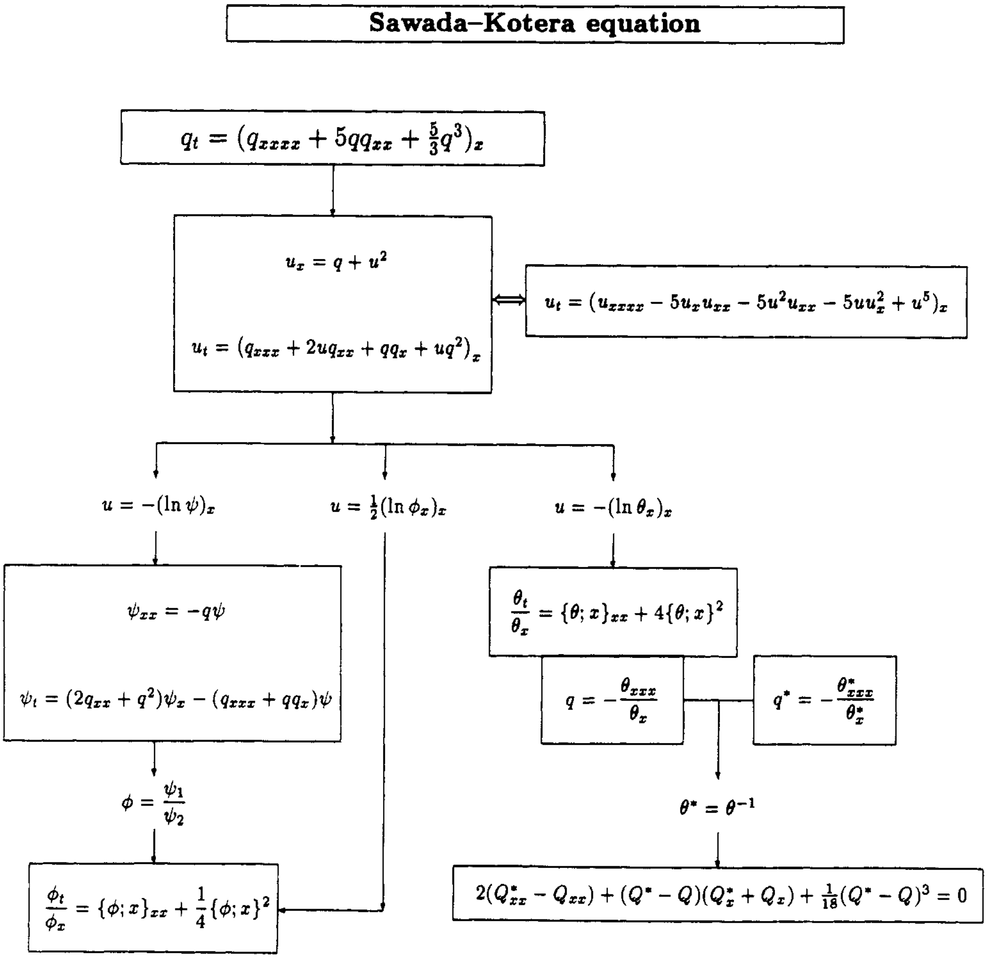

Figure 7. Properties of the Sawada–Kotera equation. Note that the singularity manifold equation is given by θ. The equation satisfied by ϕ is the singularity manifold equation of the KK equation (see figure 8).

Kaup–Kuperschmidt equation

$$q_t = (q_{xxxx} + 5qq_{xx} + \tfrac{15}{4}q_x^2 + \tfrac{5}{3}q^3)_x$$

$$u_x = -\frac{q}{2} - \frac{u^2}{2}$$

$$u_t = \left(-\frac{q_{xxx}}{2} + \frac{u}{2}q_{xx} - 2qq_x + uq^2\right)_x$$

$$\Longleftrightarrow \quad u_t = (u_{xxxx} - 5u_x u_{xx} - 5u^2 u_{xx} - 5uu_x^2 + u^5)_x$$

$$u = 2(\ln\tilde{\psi})_x \qquad u = -(\ln\theta_x)_x \qquad u = \tfrac{1}{2}(\ln\phi_x)_x$$

$$\tilde{\psi}_{xx} = -\frac{q}{4}\tilde{\psi}$$

$$\tilde{\psi}_t = \left(\frac{q_{xx}}{2} + q^2\right)\tilde{\psi}_x - \left(\frac{q_{xxx}}{4} + qq_x\right)\tilde{\psi}$$

$$\theta = \frac{\tilde{\psi}_1}{\tilde{\psi}_2}$$

$$\frac{\theta_t}{\theta_x} = \{\theta; x\}_{xx} + 4\{\theta; x\}^2$$

$$\frac{\phi_t}{\phi_x} = \{\phi; x\}_{xx} + \frac{1}{4}\{\phi; x\}^2$$

$$q = -\{\phi; x\} - \frac{3}{4}\frac{\phi_{xx}^2}{\phi_x^2} \qquad q^* = -\{\phi^*; x\} - \frac{3}{4}\frac{\phi_{xx}^{*2}}{\phi_x^{*2}}$$

$$\phi^* = \phi^{-1}$$

$$2(Q^* - Q)(Q^*_{xx} - Q_{xx}) + (Q^* - Q)^2(Q^*_x + Q_x) - \tfrac{3}{2}(Q^*_x - Q_x)^2 + \tfrac{1}{18}(Q^* - Q)^4 = 0$$

Figure 8. Properties of the Kaup–Kuperschmidt equation. Note that $\tilde{\psi}$ is the spectral function.

(Nucci 1990) and by the trivial spectral function itself ($\phi = \psi$); yet, for each of them, there exists a pseudopotential $\hat{u}$ such that (Nucci 1989):

$$\hat{u}_x = k\hat{u}^2 + \hat{F}_0(q, q_x) \tag{3.1a}$$

$$\hat{u}_t = \hat{G}(\hat{u}, q, q_x, q_{xx}, q_{xxx}, \ldots). \tag{3.1b}$$

Then, their singularity manifold equations can also be obtained by

$$\hat{u} = \frac{1}{2k}(\ln \phi_x)_x. \tag{3.2}$$

In the case of the SK equation, there exists a pseudopotential such that (2.8) is true, but the singularity manifold equation of the KK equation is obtained by either (2.7) or (2.9). In the case of the KK equation, we have an analogous cross result, because the SK and KK equations possess a common pseudopotential (Nucci 1988b). Here and throughout this paper the spectral parameter is taken equal to zero to simplify the calculations; only the spatial part of the auto-Bäcklund transformations is considered; $\{\phi; x\}$ represents the Schwarzian derivative of ϕ, i.e.

$$\{\phi; x\} = \left(\frac{\phi_{xx}}{\phi_x}\right)_x - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \tag{3.3}$$

and $Q = \int q \, \mathrm{d}x$, $Q^* = \int q^* \, \mathrm{d}x$.

Remark. In the cases of the Harry Dym, SK and KK equations, the equation satisfied by the pseudopotential is not invariant under (2.5). In Nucci (1988a), the auto-Bäcklund transformations given are actually trivial: there is a sign mistake. In the case of the Harry Dym equation, only trivial auto-Bäcklund transformations are found. A reciprocal auto-Bäcklund transformation, which involves the independent variables, was determined in Rogers and Wong (1984). Its generalisation to 2+1 dimensions is given in Rogers (1987). In the case of both SK and KK equations, the auto-Bäcklund transformations (Rogers and Carillo 1987) are obtained by the corresponding singularity manifold equations and their invariance under the Möbius group.

4. Examples in 2+1 dimensions

Here the results of §2 are applied to some non-linear evolution equations in 2+1 dimensions: KP (figure 9), Boussinesq (figure 10), modified KP (figure 11) and $(2+1)$-dimensional SK (figure 12) equations. In the case of the KP equation, the singularity manifold equation is obtained by (2.7). The Boussinesq equation is not an evolution equation in 2+1 dimensions, but it is linked to the KP equation by a transformation involving the independent variables, i.e. $t = 0$, $y = \tau$. Then the results obtained for the KP equation yield those for the Boussinesq equation. In the case of the modified KP equation, the singularity manifold equation is satisfied by the spectral function itself ($\phi = \psi$). In the case of the $(2+1)$-dimensional SK equation, the singularity manifold equation is given by the spectral function θ itself because of the third-order scattering problem.

Kadomtsev–Petviashvili equation

$$q_t = q_{xxx} - 6qq_x + 3\int q_{yy}\,\mathrm{d}x$$

$$u_x = q - u^2 - \int u_y\,\mathrm{d}x\,.$$

$$u_t = \left(q_x - 2uq - 4u_y - 4u\int u_y\,\mathrm{d}x + 3\int q_y\,\mathrm{d}x\right)_x$$

$$u_x^* = q^* - u^{*2} + \int u_y^*\,\mathrm{d}x$$

$$u_t^* = \left(q_x^* - 2u^*q^* + 4u_y^* + 4u^*\int u_y^*\,\mathrm{d}x - 3\int q_y^*\,\mathrm{d}x\right)_x$$

$$u = (\ln\psi)_x$$

$$\psi_{xx} = q\psi - \psi_y$$

$$\psi_t = -2q\psi_x + \left(q_x + 3\int q_y\,\mathrm{d}x\right)\psi - 4\psi_{xy}$$

$$u^* = -u$$

$$\phi = \frac{\psi_1}{\psi_2}$$

$$(Q^* - Q)^2 - 2(Q_x^* + Q_x) - 2\int(Q_y^* - Q_y)\,\mathrm{d}x = 0$$

$$\left[\frac{\phi_t}{\phi_x} - \{\phi;x\} - \frac{3}{2}\left(\frac{\phi_y}{\phi_x}\right)^2\right]_x - 3\left(\frac{\phi_y}{\phi_x}\right)_y = 0$$

Figure 9. Properties of the Kadomtsev–Petviashvili equation. In Morris (1976b) the differential form approach was used to generate the Lax equations.

Boussinesq equation

$$q_{TT} + \left(\frac{q_{xx}}{3} - q^2\right)_{xx} = 0$$

$$u_x = q - u^2 - \int u_T \, dx$$

$$u_T = \frac{q_x}{4} - \frac{uq}{2} - u\int u_T \, dx + \frac{3}{4}\int q_T \, dx$$

$$u = (\ln\psi)_x$$

$$\psi_T = q\psi + \psi_{xx}$$

$$\psi_{xxx} = -\frac{3}{2}q\psi_x - \frac{3}{4}\left(q_x + \int q_T \, dx\right)\psi$$

$$\phi = \frac{\psi_1}{\psi_2}$$

$$\left[\{\phi; x\} + \frac{3}{2}\left(\frac{\phi_T}{\phi_x}\right)^2\right]_x + 3\left(\frac{\phi_T}{\phi_x}\right)_T = 0$$

$$u^*_x = q^* - u^{*2} + \int u^*_T \, dx$$

$$u^*_T = -\frac{q^*_x}{4} + \frac{u^* q^*}{2} - u^*\int u^*_T \, dx + \frac{3}{4}\int q^*_T \, dx$$

$$u^* = -u$$

$$(Q^* - Q)^2 - 2(Q^*_x + Q_x) - 2\int (Q^*_T - Q_T)\, dx = 0$$

Figure 10. Properties of the Boussinesq equation. In Morris (1976a) the differential form approach was used to generate the Lax equations.

Modified Kadomtsev–Petviashvili equation

$$\boxed{q_t = q_{xxx} - 6q^2 q_x + 3\int q_{yy}\,\mathrm{d}x}$$

$$u_x = -2uq + u^2 - \int u_y\,\mathrm{d}x$$

$$u_t = \left(-2uq_x - 2uq^2 - 4u_y - 4(q-u)\int u_y\,\mathrm{d}x - 6u\int q_y\,\mathrm{d}x\right)_x$$

$$\boxed{u^*_x = -2u^*q^* + u^{*2} + \int u^*_y\,\mathrm{d}x}$$

$$u = -(\ln\psi)_x \qquad\qquad u^* = -u$$

$$\psi_{xx} = -2q\psi_x - \psi_y$$

$$\psi_t = -2\left(q_x + q^2 + 3\int q_y\,\mathrm{d}x\right)\psi_x - 4\psi_{xy} - 4q\psi_y$$

$$\Updownarrow$$

$$\boxed{\frac{\psi_t}{\psi_x} = \{\psi; x\} + 3\int\left(\frac{\psi_y}{\psi_x}\right)_y \mathrm{d}x + \frac{3}{2}\left(\frac{\psi_y}{\psi_x}\right)^2}$$

$$\boxed{(Q^*_x - Q_x) + \exp[-(Q^* + Q)] + \exp(Q^* + Q)\int(Q^*_y + Q_y)\exp[-(Q^* + Q)]\,\mathrm{d}x = 0}$$

Figure 11. Properties of the modified Kadomtsev–Petviashvili equation. Note that ψ is the spectral function which satisfies the singularity manifold equation. The equation for u^*_t is not shown.

(2+1)-dimensional Sawada–Kotera equation

$$q_t = (q_{xxxx} + 5qq_{xx} + \tfrac{5}{3}q^3 + 5q_{xy})_x - 5\int q_{yy}\,\mathrm{d}x + 5qq_y + 5q_x\int q_y\,\mathrm{d}x$$

$$\downarrow$$

$$u_x = q + u^2 - \mathrm{e}^{\int u\,\mathrm{d}x}\int\left(\int u_y\,\mathrm{d}x\right)\mathrm{e}^{-\int u\,\mathrm{d}x}\,\mathrm{d}x$$

$$u_t = \left(q_{xxx} + 2uq_{xx} + uq^2 + qq_x + 4q_y + 5u\int q_y\,\mathrm{d}x - 3q\int u_y\,\mathrm{d}x + 3q_x\,\mathrm{e}^{\int u\,\mathrm{d}x}\int\left(\int u_y\,\mathrm{d}x\right)\mathrm{e}^{-\int u\,\mathrm{d}x}\,\mathrm{d}x\right.$$
$$\left.-9\,\mathrm{e}^{\int u\,\mathrm{d}x}\int\left[\int u_{yy}\,\mathrm{d}x - \left(\int u_y\,\mathrm{d}x\right)^2\right]\mathrm{e}^{-\int u\,\mathrm{d}x}\,\mathrm{d}x\right)_x$$

$$\downarrow$$

$$u = -(\ln\theta_x)_x$$

$$\downarrow$$

$$\theta_{xxx} = -q\theta_x - \theta_y$$

$$\theta_t = -3q_x\theta_{xx} - \left(q_{xx} - q^2 - 5\int q_y\,\mathrm{d}x\right)\theta_x + 9\theta_{xxy} + 6q\theta_y$$

$$\Updownarrow$$

$$\left[\frac{\theta_t}{\theta_x} - \{\theta; x\}_{xx} - 4\{\theta; x\}^2 - 5\left(\frac{\theta_y}{\theta_x}\right)_{xx}\right]_x - 10\{\theta; x\}\left(\frac{\theta_y}{\theta_x}\right)_x + 5\left(\frac{\theta_y}{\theta_x}\right)_y = 0$$

$$q = -\frac{\theta_{xxx}}{\theta_x} - \frac{\theta_y}{\theta_x} \qquad q^* = -\frac{\theta^*_{xxx}}{\theta^*_x} - \frac{\theta^*_y}{\theta^*_x}$$

$$\downarrow$$

$$\theta^* = \theta^{-1}$$

$$\downarrow$$

$$(Q^* - Q)[2(Q^* - Q)_{xx} + Q^*_x + Q_x] + \tfrac{1}{18}(Q^* - Q)^3 + 2\int(Q^*_y - Q_y)\,\mathrm{d}x = 0$$

Figure 12. Properties of the (2+1)-dimensional Sawada–Kotera equation. Note that θ is the spectral function (third-order scattering problem) which satisfies the singularity manifold equation. The auto-Bäcklund transformation is given through the singularity manifold equation and its invariance under the Möbius group.

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